Double Covers of Symplectic Dual Polar Graphs

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Abstract

Let $\Gamma = \Gamma(2n, q)$ be the dual polar graph of type $Sp(2n, q)$. Underlying this graph is a $2n$-dimensional vector space V over a field $\mathbb{F}_q$ of odd order q, together with a symplectic (i.e. nondegenerate alternating bilinear) form $B : V \times V \to \mathbb{F}_q$. The vertex set of Γ is the set $\mathcal{V}$ of all n-dimensional totally isotropic subspaces of V. If $q \equiv 1 \bmod 4$, we obtain from Γ a nontrivial two-graph $\Delta = \Delta(2n, q)$ on $\mathcal{V}$ invariant under $PSp(2n, q)$. This two-graph corresponds to a double cover $\widehat{\Gamma} \to \Gamma$ on which is naturally defined a Q-polynomial $(2n + 1)$-class association scheme on $2|\widehat{\mathcal{V}}|$ vertices.

Keywords: association scheme, Q-polynomial, symplectic group, two-graph, dual polar graph

1. Introduction

Association schemes [2, 6] were first defined by Bose and Mesner [5] in the context of the design of experiments. Philippe Delsarte used association schemes to unify the study of coding theory and design theory in his thesis [9], where he derived his well-known linear programming bound which has since found many applications in combinatorics. There he identified two types of association schemes which were of particular interest: the so-called P-polynomial and Q-polynomial schemes. Schemes which are P-polynomial are precisely those arising from distance-regular graphs, and are well studied. In particular, much effort has gone into the classification of distance-transitive graphs, the P-polynomial schemes which are the orbitals of a permutation group; and it is likely that all such examples are known. Also well-studied are the schemes which are both Q-polynomial and P-polynomial. A well-known conjecture [2, p.312] of Bannai and Ito is the following: for sufficiently large d, a primitive scheme is P-polynomial if and only if it is Q-polynomial.

Classification efforts for Q-polynomial schemes are far less advanced than in the P-polynomial case; in particular it is likely that more examples from permutation groups are yet to be found. The Q-polynomial property has no known

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combinatorial characterization, making their study more difficult. However, the list of known examples (see [13, 15, 8]) indicates that these objects have interesting structure from the viewpoint of designs, lattices, coding theory and finite geometry.

In this paper, we give a new family of imprimitive Q-polynomial schemes with an unbounded number of classes. These schemes are formed by the orbitals of a group, giving a double cover of the scheme arising from the symplectic dual polar space graph. We note that only one other family of imprimitive Q-polynomial schemes with an unbounded number of classes is known that is not P-polynomial, namely the bipartite doubles of the Hermitian dual polar space graphs, which are Q-bipartite and Q-antipodal. The schemes in this paper are Q-bipartite, and have two Q-polynomial orderings. Except when the field order q is a square, the splitting field of these schemes is also irrational. We note that this is the only known family of Q-polynomial schemes with unbounded number of classes and an irrational splitting field. In the last section we give open parameters for hypothetical primitive Q-polynomial subschemes of this family.

Our paper is organized as follows: Background material on Gaussian coefficients, two-graphs and double covers of graphs, are covered in Sections 2–3. In Section 4 we recall the standard construction of the symplectic dual polar graph $\Gamma = \Gamma(2n, q)$. There we also introduce the Maslov index, which we use in Section 5 to construct the double cover $\widehat{\Gamma} \to \Gamma$ when $q \equiv 1 \bmod 4$. In Section 6 we construct a $(2n+1)$-class association scheme $\mathcal{S} = \mathcal{S}_{n,q}$ from $\widehat{\Gamma}$; and in Section 7 we show that $\mathcal{S}$ is Q-polynomial. The P-matrix of the scheme is constructed in Section 8. A particularly tantalizing open problem is the question whether $\mathcal{S}$ is in general the extended Q-bipartite double of a primitive Q-polynomial scheme; see Section 9.

2. Gaussian coefficients

For all integers n, k we define the *Gaussian coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}, & \text{if } k \geqslant 0; \\ 0, & \text{if } k < 0. \end{cases}$$

In particular for $k = 0$ the empty product gives $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$. In later sections, q will be a fixed prime power; but here we may regard q as an indeterminate, so that for $n \geqslant 0$, after cancelling factors we find $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}[q]$; and specializing to $q = 1$ gives the ordinary binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$. For general $n \in \mathbb{Z}$ we instead obtain a Laurent polynomial in q with integer coefficients, i.e. $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{Z}[q, q^{-1}]$, as follows from conclusion (ii) of the following.

Proposition 2.1. *Let $n, k, \ell \in \mathbb{Z}$. The Gaussian coefficients satisfy*

(i) $\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$;

(ii) $\begin{bmatrix}-n\\k\end{bmatrix} = (-q^{-n})^k \begin{bmatrix}n+k-1\\k\end{bmatrix}$*;*

(iii) $\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\\ell\end{bmatrix} = \begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}n-\ell\\k-\ell\end{bmatrix}$*;*

(iv) $\begin{bmatrix}n\\k\end{bmatrix} = \begin{bmatrix}n\\n-k\end{bmatrix}$ *whenever* $0 \leqslant k \leqslant n$*.* □

Most of the conclusions of Proposition 2.1 are found in standard references such as [1]. However, our definition of $\begin{bmatrix}n\\k\end{bmatrix}$ differs from the standard definition found in most sources, which either leave $\begin{bmatrix}n\\k\end{bmatrix}$ undefined for $n < 0$, or define it to be zero in that case. Our extension to all $n \in \mathbb{Z}$ means that the recurrence formulas (i) hold for all integers n, k, unlike the 'standard definition' which fails for $n = k = 0$. Property (i) plays a role in our later algebraic proofs using generating functions. In further defense of our definition, we observe that it has become standard to extend the definition of binomial coefficients $\binom{n}{k}$ so that $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$ (see e.g. [1, p.12]); and (ii) naturally generalizes this to Gaussian coefficients. We further note that (iii) holds for all $n, k \in \mathbb{Z}$ whether one takes the standard definition of $\begin{bmatrix}n\\k\end{bmatrix}$ or ours. The one advantage of the standard definition is that it renders superfluous the extra restriction $0 \leqslant k \leqslant n$ in the symmetry condition (iv). The interpretation of $\begin{bmatrix}n\\k\end{bmatrix}$ as the number of k-subspaces of an n-space over $\mathbb{F}_q$ is valid for all $n \geqslant 0$.

In Section 8 we will make use of the well-known generating polynomials

$$E_m(t) = \prod_{i=0}^{m-1} (1 + q^i t) = \sum_{\ell=0}^{\infty} q^{\binom{\ell}{2}} \begin{bmatrix}m\\\ell\end{bmatrix} t^\ell \quad \text{for } m = 0, 1, 2, \ldots;$$

note that in the latter sum, the terms for $\ell > m$ vanish, yielding $E_m(t) \in \mathbb{Z}[q, t]$ (or after specializing to a fixed prime power q, we obtain $E_m(t) \in \mathbb{Z}[t]$). Here we see the usual binomial coefficient $\binom{\ell}{2} = \frac{1}{2}\ell(\ell - 1)$. In Section 8 we will make use of the following obvious relations:

Proposition 2.2. *For all* $m \geqslant 0$*, the generating function* $E_m(t)$ *satisfies*

(i) $E_m(-qt) = \frac{1-q^m t}{1-t} E_m(-t)$*;*

(ii) $E_m(q^2 t) = \frac{1+q^{m+1} t}{1+qt} E_m(qt)$*; and*

(iii) $E_m(r^3 t) = \frac{1+rq^m t}{1+rt} E_m(rt)$ *where* $r = \sqrt{q}$*.* □

3. Two-graphs and double covers of graphs

Here we describe the most basic connections between two-graphs and double covers of graphs; see [14, 16, 6, 18] for more details. Our notation is chosen to conform to that used in subsequent sections.

Let $\mathcal{V}$ be any set. Denote by $\binom{\mathcal{V}}{k}$ the collection of all k-subsets of $\mathcal{V}$ (i.e. subsets of cardinality k). A *two-graph on* $\mathcal{V}$ is a subset $\Delta \subseteq \binom{\mathcal{V}}{3}$ such that for every 4-set $\{x, y, z, w\} \in \binom{\mathcal{V}}{4}$, an even number, i.e. 0, 2 or 4, of the triples

$\{x,y,z\}$, $\{x,y,w\}$, $\{x,z,w\}$, $\{y,z,w\}$ is in Δ. If Δ is a two-graph on $\mathcal{V}$, then the complementary set of triples $\overline{\Delta} = \big\{\{x,y,z\} \in \binom{\mathcal{V}}{3} : \{x,y,z\} \notin \Delta\big\}$ is also a two-graph, called the *complementary two-graph on* $\mathcal{V}$.

A *graph* on $\mathcal{V}$ is a subset $\Gamma \subseteq \binom{\mathcal{V}}{2}$. Elements of Γ are called *edges*. The *complete graph on* $\mathcal{V}$ is the graph $K_{\mathcal{V}}$ with full edge set $\binom{\mathcal{V}}{2}$. In general the complementary set of pairs $\overline{\Gamma} = \big\{\{x,y\} \in \binom{\mathcal{V}}{2} : \{x,y\} \notin \Gamma\big\}$ is the *complementary graph on* $\mathcal{V}$.

Every graph on $\mathcal{V}$ may be identified with a signing of the edges of the complete graph $K_{\mathcal{V}}$, i.e. a function $\sigma : \binom{\mathcal{V}}{2} \to \{\pm1\}$. Under this correspondence, the graph corresponding to σ has as its edge set $\sigma^{-1}(1) = \big\{\{x,y\} \in \binom{\mathcal{V}}{2} : \sigma(x,y) = 1\big\}$. (Here we abbreviate $\sigma(\{x,y\}) = \sigma(x,y)$.)

Given Γ and σ as above (which amounts to two graphs which may be entirely unrelated except for sharing the same vertex set $\mathcal{V}$), we construct a new graph $\widehat{\Gamma} = \widehat{\Gamma}_\sigma$ with vertex set $\widehat{\mathcal{V}} = \mathcal{V} \times \{\pm1\}$ and adjacency relation defined by

$$(x,\varepsilon) \sim (y,\varepsilon') \iff x \sim y \text{ and } \varepsilon\varepsilon' = \sigma(x,y).$$

(Note that $(x,1) \not\sim (x,-1)$ since Γ has no loops.) The map $(x,\varepsilon) \mapsto x$ is a *double covering map* $\theta : \widehat{\Gamma} \to \Gamma$, also called a *double cover* or simply a *cover*; and the *fibers* of this map are the pairs $\theta^{-1}(x) = \{(x,1),(x,-1)\}$ where $x \in \mathcal{V}$. (By definition, a *covering map* of graphs is a graph homomorphism $\theta : \widehat{\Gamma} \to \Gamma$ such that for any vertex $x \in \Gamma$, the preimage of the neighborhood graph Γ_x is isomorphic to a disjoint union of copies of Γ_x; see e.g. [10]. 'Double' refers to the condition that the covering map is 2-to-1.) We also say that the vertices $(x,1)$ and $(x,-1)$ are *antipodal* with respect to the covering map. (Note that antipodal vertices must be at distance $\geqslant 2$; but we deviate from common custom by *not requiring* pairs of antipodal vertices to be at maximal distance $\operatorname{diam}\widehat{\Gamma}$.) We denote by ζ the transposition interchanging antipodal vertices: $(x,1) \overset{\zeta}{\leftrightarrow} (x,-1)$. Denote by $\operatorname{Aut}_\zeta \widehat{\Gamma} \leqslant \operatorname{Aut}\widehat{\Gamma}$ the subgroup consisting of all automorphisms of the graph $\widehat{\Gamma}$ which preserve the antipodality relation. In general, $\operatorname{Aut}_\zeta \widehat{\Gamma}$ is the centralizer of ζ in the full automorphism group $\operatorname{Aut}\widehat{\Gamma} \leqslant \operatorname{Sym}\widehat{\mathcal{V}}$; but in our case we obtain equality $\operatorname{Aut}_\zeta \widehat{\Gamma} = \operatorname{Aut}\widehat{\Gamma}$ (see Lemma 5.4). Similarly, two covers $\theta_i : \widehat{\Gamma}_i \to \Gamma$ of the same graph Γ (for $i = 1,2$) are *equivalent* or *isomorphic* if there is a graph isomorphism $\rho : \widehat{\Gamma}_1 \to \widehat{\Gamma}_2$ which preserves antipodality, i.e. $\theta_1 \circ \rho = \theta_2$.

Given $\sigma : \binom{\mathcal{V}}{2} \to \{\pm1\}$ as above, for every triple $\{x,y,z\} \in \binom{\mathcal{V}}{3}$ we may define

$$\sigma(x,y,z) = \sigma(x,y)\sigma(y,z)\sigma(z,x) \in \{\pm1\}.$$

A triple $\{x,y,z\} \in \binom{\mathcal{V}}{3}$ is called *coherent* or *non-coherent* according as $\sigma(x,y,z) = 1$ or -1. The set of all coherent triples forms a two-graph on $\mathcal{V}$, denote by Δ_σ; and the set of non-coherent triples gives the complementary two-graph $\overline{\Delta}_\sigma$.

Two sign functions $\sigma_1, \sigma_2 : \binom{\mathcal{V}}{2} \to \{\pm1\}$ (or the corresponding graphs $\sigma_1^{-1}(1)$, $\sigma_2^{-1}(1)$ on $\mathcal{V}$) are *switching-equivalent* in the sense of Seidel [16] if there exists a map $f : \mathcal{V} \to \{\pm1\}$ such that $\sigma_2(x,y) = f(x)f(y)\sigma_1(x,y)$ for all $\{x,y\} \in \binom{\mathcal{V}}{2}$.

We have $\Delta_{\sigma_1} = \Delta_{\sigma_2}$ iff σ_1 and σ_2 are switching-equivalent. Assuming this holds, then the corresponding covers $\widehat{\Gamma}_{\sigma_1}$ and $\widehat{\Gamma}_{\sigma_2}$ are isomorphic via $(x, \varepsilon) \mapsto (x, f(x)\varepsilon)$.

In the special case of the complete graph $\Gamma = K_{\mathcal{V}}$, the following three notions are equivalent (see [6, §1.5]): two-graphs on $\mathcal{V}$, switching classes of graphs on $\mathcal{V}$, and isomorphism classes of double covers of the complete graph $K_{\mathcal{V}}$. For example given a double cover $\widehat{K_{\mathcal{V}}} \to K_{\mathcal{V}}$, the corresponding two-graph is obtained as follows (see [16, p.488]): Each triple $\{x, y, z\}$ of distinct vertices in $\mathcal{V}$ induces a triangle $K_{\{x,y,z\}} \subseteq K_{\mathcal{V}}$; and such a triple is coherent iff its preimage in $\widehat{K_{\mathcal{V}}}$ induces a pair of triangles, rather than a 6-cycle, in $\widehat{K}_{\mathcal{V}}$.

An *automorphism of a two-graph* Δ is a permutation of the underlying point set $\mathcal{V}$ which preserves the set of coherent triples. We now relate $\operatorname{Aut}\Delta$ to the group $\operatorname{Aut}_\zeta \widehat{K} \leqslant \operatorname{Aut}\widehat{K}$ defined above for the associated double cover $\widehat{K} \to K$, where we abbreviate the complete graph $K_{\mathcal{V}} = K$. The following is easy to verify (or see [18, §2], where this isomorphism is denoted $\widehat{G}/Z \cong G$):

Proposition 3.1. *The group* $\operatorname{Aut}_\zeta \widehat{K}$ *acts naturally on* Δ*, inducing the full automorphism group of* Δ*. The kernel of this action is the central subgroup* $\langle\zeta\rangle$ *of order 2; thus* $(\operatorname{Aut}_\zeta \widehat{K})/\langle\zeta\rangle \cong \operatorname{Aut}\Delta$.

4. Dual polar graphs of type $Sp(2n, q)$, q odd

Fix a finite field $\mathbb{F}_q$ of odd prime power order q; an integer $n \geqslant 1$; a $2n$-dimensional vector space V over $\mathbb{F}_q$; and a symplectic (i.e. nondegenerate alternating) bilinear form $B : V \times V \to \mathbb{F}_q$. The *symplectic group* $Sp(2n, q)$ consists of all *(linear) isometries* of B, i.e.

$$Sp(2n, q) = \{g \in GL(V) : B(x^g, y^g) = B(x, y) \text{ for all } x, y \in V\}.$$

The group of all *(linear) similarities* of B is

$$\begin{aligned} GSp(2n, q) = \{g \in GL(V) : &\text{ for some nonzero } \mu \in \mathbb{F}_q \text{ we have} \\ &B(x^g, y^g) = \mu B(x, y) \text{ for all } x, y \in V\}; \end{aligned}$$

some other notations for this group are $GSp_n(q)$ in [12] or $CSp_n(q)$ in [4, p.31]. Replacing $GL(V)$ by $\Gamma L(V) \cong GL(V) \rtimes \operatorname{Aut}\mathbb{F}_q$, the group of all semilinear transformations of V, we obtain the group $\Sigma Sp(2n, q)$ of all *semi-isometries*, and the group $\Gamma Sp(2n, q)$ of all *semi-similarities* of B, given by

$$\begin{aligned} \Sigma Sp(2n, q) &= \{g \in \Gamma L(V) : \text{for some } \tau \in \operatorname{Aut}\mathbb{F}_q \text{ we have} \\ &\qquad\qquad B(x^g, y^g) = B(x, y)^\tau \text{ for all } x, y \in V\} \\ &\cong Sp(2n, q) \rtimes \operatorname{Aut}\mathbb{F}_q; \\ \Gamma Sp(2n, q) &= \{g \in \Gamma L(V) : \text{for some nonzero } \mu \in \mathbb{F}_q \text{ and } \tau \in \operatorname{Aut}\mathbb{F}_q \\ &\qquad\qquad \text{we have } B(x^g, y^g) = \mu B(x, y)^\tau \text{ for all } x, y \in V\} \\ &\cong GSp(2n, q) \rtimes \operatorname{Aut}\mathbb{F}_q. \end{aligned}$$

The projective versions of these groups are

$$\begin{aligned} PSp(2n,q) &= Sp(2n,q)/\langle -I\rangle,\\ PGSp(2n,q) &= GSp(2n,q)/Z,\\ P\Sigma Sp(2n,q) &= \Sigma Sp(2n,q)/\langle -I\rangle,\\ P\Gamma Sp(2n,q) &= \Gamma Sp(2n,q)/Z \end{aligned}$$

where the central subgroup Z of order $q-1$ consists of all scalar transformations $v \mapsto \lambda v$ for $0 \neq \lambda \in \mathbb{F}_q$. We have

$$[P\Gamma Sp(2n,q) : P\Sigma Sp(2n,q)] = [PGSp(2n,q) : PSp(2n,q)] = 2$$

where the nontrivial coset in both cases is represented by $h \in GSp(2n,q)$ satisfying $B(u^h,v^h) = \eta B(u,v)$ and $\eta \in \mathbb{F}_q$ is a nonsquare.

Our choice of notation for these groups, while not universal, is intended to conform reasonably with [7, 12]. The group $P\Gamma Sp(2n,q)$, for example, is denoted $PC\Gamma Sp_n(q)$ in [4, p.31]. It arises (see Theorem 4.1) as the full automorphism group of the associated dual polar graph, which we now describe.

Denote by $\mathcal{V}$ be the collection of all maximal totally isotropic subspaces with respect to B, i.e.

$$\mathcal{V} = \{X \leqslant V : X^\perp = X\}$$

where by definition $X^\perp = \{v \in V : B(x,v) = 0 \text{ for all } x \in X\}$. Members of $\mathcal{V}$ are often called *generators*, and every $X \in \mathcal{V}$ has dimension n. Denote by $\Gamma = \Gamma(2n,q)$ the graph on $\mathcal{V}$ where two vertices $X,Y \in \mathcal{V}$ are adjacent iff $X \cap Y$ has codimension 1 in both X and Y. More generally, the distance between X and Y in Γ is $d(X,Y) = k \in \{0,1,2,\ldots,n\}$ where the subspace $X \cap Y$ has codimension k in both X and Y. Let Γ_k denote the graph of the distance-k relation on $\mathcal{V}$; i.e. Γ_k has vertex set $\mathcal{V}$ and two vertices $X,Y \in \mathcal{V}$ are adjacent in Γ_k iff $d(X,Y) = k$. The graph $\Gamma_1 = \Gamma$ is called the *dual polar graph of type* Sp$(2n,q)$. It is *distance regular*: given any two vertices X,Y in Γ at distance $k \in \{0,1,2,\ldots,n\}$, the vertex Y has $q^{\binom{n-k}{2}}\left[{n \atop k}\right]$ neighbors Z in Γ, of which

$$\begin{aligned} a_k &= q^k - 1 \text{ are at distance } k \text{ from } X,\\ b_k &= q^{k+1}\left[{n-k \atop 1}\right] \text{ are at distance } k+1 \text{ from } X, \text{ and}\\ c_k &= \left[{k \atop 1}\right] \text{ are at distance } k-1 \text{ from } X; \end{aligned}$$

see [6, §9.4]. The edges of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ partition the non-identical pairs on $\mathcal{V}$, viewed as the edges of the complete graph $K_{\mathcal{V}}$; and together with the identity relation $\Gamma_0 = \{(X,X) : X \in \mathcal{V}\}$ we obtain an n-class association scheme on $\mathcal{V}$ (see Section 6). This scheme is P-polynomial since Γ is distance regular; see [6].

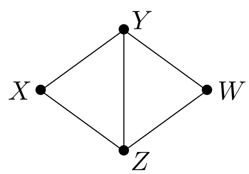
Theorem 4.1. *For $n \geqslant 2$, the full automorphism group of $\Gamma = \Gamma(2n,q)$ is the group $P\Gamma Sp(2n,q)$ acting naturally on the projective space of V.*

Proof. See [6, p.275] (where this group is however denoted $P\Sigma p(2n,q)$). $\square$

Note that when $n = 1$, the dual polar graph $\Gamma(2, q)$ is simply the complete graph K_{q+1}, whose full automorphism group is the symmetric group of degree $q + 1$.

For use in Section 5 we record the following well-known fact. Although it follows easily from the axioms of polar geometry (or of near polygons), in the interest of self-containment we include a proof.

Lemma 4.2. *The 'diamond' graph (as shown) is not an induced subgraph of the dual polar graph* Γ.



Proof. If X, Y, Z are mutually adjacent as shown, then $X \cap Y$ and $X \cap Z$ are distinct subspaces of codimension 1 in X, so $X = (X \cap Y) + (X \cap Z)$, whence $X \subseteq Y + Z$. Thus $X = X^\perp \supseteq Y^\perp \cap Z^\perp = Y \cap Z$. Similarly, $W \supseteq Y \cap Z$. Now $X \cap W$ contains a subspace of dimension $n-1$, contradicting $d(X, W) \geqslant 2$. $\square$

Now let X be any n-dimensional vector space over $\mathbb{F}_q$. An n-linear form $f : X^n \to \mathbb{F}_q$ (i.e. linear in each argument whenever the other $n-1$ arguments are fixed) is *alternating* if $f(x_1, x_2, \ldots, x_n) = 0$ whenever two x_i's coincide; equivalently, $f(x_{1^\tau}, x_{2^\tau}, \ldots, x_{n^\tau}) = -f(x_1, x_2, \ldots, x_n)$ for every odd permutation τ of the indices. The space of all such alternating forms is one-dimensional, and is canonically identified with $(\bigwedge^n X)^*$, the dual space of $\bigwedge^n X$. A *determinant function on* X is any nonzero alternating form $X^n \to \mathbb{F}_q$. Since $\dim(\bigwedge^n X)^* = 1$, a determinant function is determined up to nonzero scalar multiple.

Fix a choice of determinant function δ_X for each $X \in \mathcal{V}$. Although these choices are not canonical, one may proceed by arbitrarily choosing a basis $\psi_1, \psi_2, \ldots, \psi_n$ for $X^* = \mathrm{Hom}(X, \mathbb{F}_q)$; then we obtain a determinant function on X by defining

$$\delta_X(x_1, x_2, \ldots, x_n) = \det\bigl(\psi_i^*(x_j) : 1 \leqslant i, j \leqslant n\bigr).$$

We need to define $\sigma(X, Y) \in \{\pm 1\}$ for any pair $X \neq Y$ in $\mathcal{V}$. Let $k \in \{1, 2, \ldots, n\}$ be the codimension of $X \cap Y$ in both X and Y. Choose bases $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ for X and Y respectively, such that $x_i = y_i$ (for $k < i \leqslant n$) is a common basis for $X \cap Y$. (These bases depend on the choice of pair (X, Y) and so are unrelated to any bases for X and Y used as a crutch for constructing the corresponding determinant functions). Define

$$\sigma(X, Y) = \chi\bigl(\delta_X(x_1, x_2, \ldots, x_n)\delta_Y(y_1, y_2, \ldots, y_n) \det\bigl[B(x_i, y_j) : 1 \leqslant i, j \leqslant k\bigr]\bigr)$$

where $\chi : \mathbb{F}_q^\times \to \{\pm 1\}$ is the quadratic character: $\chi(a) = 1$ or -1 according as $a \in \mathbb{F}_q^\times$ is a square or a nonsquare. This definition is implicit in [19, 11]; and inspired by the literature, we refer to $\sigma(X, Y)$ (or the ternary function $\sigma(X, Y, Z)$

defined below) as the *Maslov index*. Note that B induces a nondegenerate bilinear form on the $2k$-space $(X+Y)/(X\cap Y)$, so that the $k\times k$ matrix $\big[B(x_i,y_j):1\leqslant i,j\leqslant k\big]$ is nonsingular.

Proposition 4.3. *Let $X,Y\in\mathcal{V}$ at distance $d(X,Y)=k\in\{0,1,2,\dots,n\}$.*

(i) The value of $\sigma(X,Y)$ is independent of the choice of bases x_i and y_j as above.

(ii) Its dependence on the choice of determinant functions is expressed as follows: Replacing δ_X by $c\delta_X$ has the effect of multiplying $\sigma(X,Y)$ by $\chi(c)$.

(iii) $\sigma(Y,X)=\chi(-1)^k\sigma(X,Y)=(-1)^{k(q-1)/2}\sigma(X,Y)$.

(iv) Let $g\in\Gamma Sp(2n,q)$, so that there exists a nonzero scalar $\mu_g\in\mathbb{F}_q$ and $\tau_g\in\operatorname{Aut}\mathbb{F}_q$ satisfying $B(x^g,y^g)=\mu_gB(x,y)^{\tau_g}$ for all $x,y\in Y$. Then there exist nonzero scalars $\lambda_{g,U}\in\mathbb{F}_q$ for $U\in\mathcal{V}$, such that

$$\sigma(X^g,Y^g)=\chi\big(\mu_g^k\lambda_{g,X}\lambda_{g,Y}\big)\sigma(X,Y).$$

Proof. Consider a change of basis on $X\cap Y$ specified by $x_i'=y_i'=\sum_{k<j\leqslant n}a_{ij}x_j$ where $A=\big(a_{ij}:k<i,j\leqslant n\big)$ is any invertible $(n-k)\times(n-k)$ matrix. Then

$$\delta_X(x_1,\dots,x_k,x_{k+1}',\dots,x_n')=(\det A)\delta_X(x_1,\dots,x_n)$$

and $\delta_Y(y_1,\dots,y_n)$ is multiplied by the same factor, $\det A$. The $(n-k)\times(n-k)$ matrix $[B(x_i,y_j):k<i,j\leqslant n]$ is unchanged, so the value of $\sigma(X,Y)$ is multiplied by a net factor of $\chi\big((\det A)^2\big)=1$.

Next consider replacing $x_1,\dots,x_k$ by $x_1',\dots,x_k'$ where

$$x_i'\equiv\sum_{1\leqslant j\leqslant k}a_{ij}x_j\mod(X\cap Y)$$

for $i=1,2,\dots,k$ where A is an invertible $k\times k$ matrix, and we leave the basis of Y unchanged. Then

$$\begin{gathered}\delta_X(x_1',\dots,x_k',x_{k+1},\dots,x_n)=(\det A)\delta_X(x_1,\dots,x_n);\\ \det\big[B(x_i',y_j):1\leqslant i,j\leqslant k\big]=(\det A)\det\big[B(x_i,y_j):1\leqslant i,j\leqslant k\big]\end{gathered}$$

and the δ_Y factor is unchanged; so once again, the value of $\sigma(X,Y)$ is multiplied by $\chi\big((\det A)^2\big)=1$. The same argument applies if $y_1,\dots,y_k$ are replaced by $y_1',\dots,y_k'$, and so (i) follows. Conclusion (ii) is clear.

Interchanging X and Y has the effect of interchanging the δ_X and δ_Y factors, and replacing

$$\begin{aligned}&\big[B(x_i,y_j):1\leqslant i,j\leqslant k\big]\mapsto\\ &\qquad\qquad\big[B(y_i,x_j):1\leqslant i,j\leqslant k\big]=-\big[B(x_i,y_j):1\leqslant i,j\leqslant k\big].\end{aligned}$$

The determinant of this matrix accrues a factor of $(-1)^k$, whence (iii) holds.

Let $g \in \Gamma Sp(2n,q)$. There exists a nonzero $\mu_g \in \mathbb{F}_q$ and $\tau_g \in \operatorname{Aut}\mathbb{F}_q$ such that $(au+bv)^g = a^{\tau_g}u^g + b^{\tau_g}v^g$ and $B(u^g,v^g) = \mu_g B(u,v)^{\tau_g}$ for all $a,b \in \mathbb{F}_q$ and $u,v \in V$. Now the map

$$X^n \to \mathbb{F}_q, \quad (x_1,x_2,\ldots,x_n) \mapsto \delta_{X^g}\big(x_1^g,x_2^g,\ldots,x_n^g\big)^{\tau_g^{-1}}$$

is a determinant function on X, so it is a scalar multiple of $\delta_X(x_1,x_2,\ldots,x_n)$. Hence there exists a nonzero scalar $\lambda_X = \lambda_{g,X} \in \mathbb{F}_q$ such that

$$\delta_{X^g}\big(x_1^g,x_2^g,\ldots,x_n^g\big) = \lambda_{g,X}\delta_X(x_1,x_2,\ldots,x_n)^{\tau_g}$$

for all $x_1,x_2,\ldots,x_n \in X$.

Now given $X,Y \in \mathcal{V}$ at distance k, fix bases x_i,y_i as before; then

$$\begin{aligned}
\sigma(X^g,Y^g) &= \chi\big(\delta_{X^g}(x_1^g,x_2^g,\ldots,x_n^g)\delta_{Y^g}(y_1^g,y_2^g,\ldots,y_n^g) \\
&\qquad\qquad \times \det\big[B(x_i^g,y_j^g) : 1 \leqslant i,j \leqslant k\big]\big) \\
&= \chi\big(\lambda_{g,X}\delta_X(x_1,x_2,\ldots,x_n)^{\tau_g}\lambda_{g,Y}\delta_Y(y_1,y_2,\ldots,y_n)^{\tau_g} \\
&\qquad\qquad \times \det\big[\mu_g B(x_i,y_j)^{\tau_g} : 1 \leqslant i,j \leqslant k\big]\big) \\
&= \chi\big(\mu_g^k\lambda_{g,X}\lambda_{g,Y}\big)\chi\big(\delta_X(x_1,x_2,\ldots,x_n)^{\tau_g}\delta_Y(y_1,y_2,\ldots,y_n)^{\tau_g} \\
&\qquad\qquad \times \det\big[B(x_i,y_j) : 1 \leqslant i,j \leqslant k\big]^{\tau_g}\big) \\
&= \chi\big(\mu_g^k\lambda_{g,X}\lambda_{g,Y}\big)\sigma(X,Y)
\end{aligned}$$

since $\chi(a^\tau) = \chi(a)$. This proves (iv). □

For each triple (X,Y,Z) with distinct $X,Y,Z \in \mathcal{V}$, define

$$\sigma(X,Y,Z) = \sigma(X,Y)\sigma(Y,Z)\sigma(Z,X) \in \{\pm 1\}.$$

A triple (X,Y,Z) of distinct elements of $\mathcal{V}$ is *coherent* or *non-coherent* according as $\sigma(X,Y,Z) = 1$ or -1.

Theorem 4.4. *Suppose $q \equiv 1 \mod 4$. Then the set of coherent triples forms a two-graph Δ_σ on $\mathcal{V}$, invariant under $P\Sigma Sp(2n,q)$.*

Proof. Let $X,Y,Z,W \in \mathcal{V}$ be distinct. Since $\chi(-1) = 1$, (X,Y,Z) is coherent iff any permutation of its members yields a coherent triple; so the set of coherent triples may be regarded as a collection of unordered triples $\{X,Y,Z\}$. Since

$$\begin{aligned}
&\sigma(X,Y,Z)\sigma(X,Y,W)\sigma(X,Z,W)\sigma(Y,Z,W) \\
&\qquad\qquad\qquad\qquad = \sigma(X,Y)^2\sigma(X,Z)^2\cdots\sigma(Z,W)^2 = 1,
\end{aligned}$$

evenly many of the triples in $\{X,Y,Z,W\}$ are coherent. If $g \in \Gamma Sp(2n,q)$ with $B(x^g,y^g) = \mu_g B(x,y)^{\tau_g}$, then

$$\begin{aligned}
\sigma(X^g,Y^g,Z^g) &= \chi(\mu_g^{d(X,Y)}\lambda_{g,X}\lambda_{g,Y})\chi(\mu_g^{d(Y,Z)}\lambda_{g,Y}\lambda_{g,Z}) \\
&\quad \times \chi(\mu_g^{d(Z,X)}\lambda_{g,Z}\lambda_{g,X})\sigma(X,Y,Z) \\
&= \chi(\mu_g)^{d(X,Y)+d(Y,Z)+d(Z,X)}\sigma(X,Y,Z).
\end{aligned}$$

In particular when $g \in \Sigma Sp(2n,q)$, $\mu_g = 1$ and $\sigma(X^g,Y^g,Z^g) = \sigma(X,Y,Z)$. □

If $q \equiv 3 \mod 4$, or $g \in P\Gamma Sp(2n, q)$ with $g \notin P\Sigma Sp(2n, q)$, the situation is a little trickier: various subsets of the coherent triples form either two-graphs or skew two-graphs in the sense of [14], invariant under $Sp(2n, q)$. We ignore this case here, and *henceforth assume that*

$$q \equiv 1 \mod 4.$$

We next show that *in a geodesic path, every triple of vertices is coherent.*

Lemma 4.5. *Suppose $q \equiv 1 \mod 4$. Let $X, Y, Z \in \mathcal{V}$ such that $d(X, Y) = j$, $d(Y, Z) = k - j$ and $d(X, Z) = k$ where $1 \leqslant j < k \leqslant n$. Then $\sigma(X, Y, Z) = 1$.*

Proof. Choose a hyperbolic basis $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_n$ for V, so that $B(e_i, e_j) = B(f_i, f_j) = 0$ and $B(e_i, f_j) = \delta_{ij}$. Since $\mathrm{Sp}(2n, q)$ is transitive on triples of generators satisfying the given distance constraints, by Theorem 4.4 we may suppose that

$$X = \langle e_1, e_2, \dots, e_n \rangle, \quad Y = \langle f_1, f_2, \dots, f_j, e_{j+1}, e_{j+2}, \dots, e_n \rangle,$$
$$Z = \langle f_1, f_2, \dots, f_k, e_{k+1}, e_{k+2}, \dots, e_n \rangle.$$

We choose the determinant function δ_X on X given by

$$\delta_X(x_1, x_2, \dots, x_n) = \det\big[B(x_i, f_j) \,:\, 1 \leqslant i, j \leqslant n\big] \quad \text{for } x_1, x_2, \dots, x_n \in X;$$

this is nothing other than the determinant of the $n \times n$ matrix whose columns are the coordinates of $x_1, \dots, x_n$ with respect to the basis $e_1, e_2, \dots, e_n$. The determinant functions δ_Y, δ_Z on Y and on Z are defined similarly, using the bases for Y and on Z listed above. The computation of $\sigma(X, Z)$ is simplified by the fact that a basis for $X \cap Z$ is $e_{k+1}, e_{k+2}, \dots, e_n$. We have

$$\delta_X(e_1, e_2, \dots, e_n) = \delta_Z(f_1, \dots, f_k, e_{k+1}, \dots, e_n) = 1$$

and $\big[B(e_i, f_j) \,:\, 1 \leqslant i, j \leqslant k\big]$ is a $k \times k$ identity matrix, with determinant 1; thus $\sigma(X, Z) = 1$. Exactly the same reasoning gives $\sigma(X, Y) = \sigma(Y, Z) = 1$, so $\sigma(X, Y, Z) = 1$. □

In the case of triples X, Y, Z not lying on geodesic paths, however, σ (or its two-graph Δ_σ) yields interesting nontrivial information. In particular, the restriction of Δ_σ to partial spreads (sets of vertices of Γ mutually at distance n) was investigated in [14, §6]. Here we consider triangles in Γ:

Lemma 4.6. *Suppose $q \equiv 1 \mod 4$. Let $X, Y \in \mathcal{V}$ such that $d(X, Y) = 1$, i.e. X and Y are adjacent in Γ. There are $a_1 = q-1$ common neighbors Z of X and Y in Γ; and exactly half of the resulting triples (X, Y, Z) are coherent.*

Proof. Choose a hyperbolic basis e_i, f_i as in the proof of Lemma 4.5. Again without loss of generality,

$$X = \langle e_1, e_2, \dots, e_n \rangle, \quad Y = \langle f_1, e_2, \dots, e_n \rangle, \quad Z = \langle e_1 + \alpha f_1, e_2, \dots, e_n \rangle$$

where $0 \neq \alpha \in \mathbb{F}_q$. The $q-1$ choices of α give exactly the $q-1$ common neighbors of X and Y in Γ. These bases of X, Y, Z give rise to natural choices of determinant functions δ_X, δ_Y, δ_Z as described in the proof of Lemma 4.5. When computing $\sigma(X,Y), \sigma(Y,Z), \sigma(Z,X)$, we use $e_2, e_3, \dots, e_n$ as the basis of $X \cap Y = X \cap Z = Y \cap Z$. Now

$$\delta_X(e_1, e_2, \dots, e_n) = \delta_Z(e_1+\alpha f_1, e_2, \dots, e_n) = 1$$

and $B(e_1, e_1+\alpha f_1) = \alpha$, so $\sigma(X,Z) = \chi(\alpha)$. Similarly, $\sigma(X,Y) = \sigma(Y,Z) = 1$ and

$$\sigma(X,Y,Z) = \chi(\alpha).$$

Since exactly half the nonzero elements of $\mathbb{F}_q$ are squares, the result follows. $\square$

Theorem 4.7. *Suppose $q \equiv 1 \mod 4$. Let $X, Y \in \mathcal{V}$ such that $d(X,Y) = k$. Then Y has exactly $a_k = q^k-1$ neighbors $Z \in \mathcal{V}$ at distance k from X in Γ; and exactly half of the resulting triples (X,Y,Z) are coherent.*

Proof. The result holds for $k=1$ by Lemma 4.6, so we may assume $k \geqslant 2$. Given $X, Y \in \mathcal{V}$ with $d(X,Y) = k$, there are $\left[{k \atop 1}\right]$ choices of hyperplane $H < Y$ containing $X \cap Y$. Each such H yields an $\mathrm{Sp}(2,q)$-space $H^\perp/H$, which contains $q+1$ subspaces of the form Z/H with $Z \in \mathcal{V}$. One such Z has distance $k-1$ from X, this being the subspace $W = (Y \cap Z) + X \cap (Y+Z) = (Y+Z) \cap (X+(Y \cap Z))$. If we exclude W and Y itself, this leaves exactly $q-1$ choices of Z having the required distances from X and Y; and this gives $(q-1)\left[{k \atop 1}\right] = q^k-1 = a_k$ choices of Z, the full number. But for how many such Z is the resulting triple (X,Y,Z) coherent? In each case $\sigma(X,W,Y) = \sigma(X,W,Z) = 1$ by Lemma 4.5; therefore $\sigma(X,Y,Z) = \sigma(W,Y,Z)$. But by Lemma 4.6, given W, Y at distance 1, exactly half of the $q-1$ choices of Z yield coherent triples (W,Y,Z). Therefore among the $a_k = (q-1)\left[{k \atop 1}\right]$ triples (X,Y,Z) with fixed X and Y, exactly $\frac{q-1}{2}\left[{k \atop 1}\right] = (q^k-1)/2$ such triples are coherent. $\square$

5. The Double Cover $\widehat{\Gamma} \to \Gamma$

The resulting double cover $\widehat{\Gamma} = \widehat{\Gamma}(2n,q) \to \Gamma(2n,q)$ has vertex set $\widehat{\mathcal{V}} = \mathcal{V} \times \{\pm 1\}$ and adjacency relation

$$(X,\varepsilon) \sim (Y,\varepsilon') \qquad \Longleftrightarrow \qquad d(X,Y) = 1 \text{ and } \varepsilon\varepsilon' = \sigma(X,Y).$$

The covering map is given by $(X,\varepsilon) \mapsto X$.

Theorem 5.1. *Every geodesic path*

$$X_0 \sim X_1 \sim \cdots \sim X_k$$

in Γ (meaning that $d(X_i, X_j) = |j-i|$) lifts to exactly two paths

$$(X_0, \varepsilon_0) \sim (X_1, \varepsilon_1) \sim \cdots \sim (X_k, \varepsilon_k)$$

in $\widehat{\Gamma}$, in which $\varepsilon_k = \varepsilon_0 \sigma(X_0, X_k)$ for each $k \geqslant 1$; thus any one of the ε_i determines all the others along this path.

Proof. We have $\varepsilon_1 = \varepsilon_0\sigma(X_0, X_1)$ by definition of adjacency in $\widehat{\Gamma}$. Assuming that $\varepsilon_i = \varepsilon_0\sigma(X_0, X_i)$ for some $i \in \{1, 2, \ldots, k-1\}$,

$$\varepsilon_{i+1} = \varepsilon_i\sigma(X_i, X_{i+1}) = \varepsilon_0\sigma(X_0, X_i)\sigma(X_i, X_{i+1}) = \varepsilon_0\sigma(X_0, X_{i+1})$$

since $\sigma(X_0, X_i, X_{i+1}) = 1$ by Lemma 4.5. $\square$

However, not every geodesic path in $\widehat{\Gamma}$ is obtained by lifting a geodesic path in Γ. For example if (X, Y, Z) is an incoherent triangle in Γ, say with $\sigma(X, Y) = \varepsilon$, $\sigma(Y, Z) = \varepsilon'$ and $\sigma(X, Z) = -\varepsilon\varepsilon'$, then

$$(X, 1) \sim (Y, \varepsilon) \sim (Z, \varepsilon\varepsilon') \sim (X, -1)$$

is a geodesic path of length 3 in $\widehat{\Gamma}$, obtained by lifting a closed path of length 3 (not a geodesic path) in Γ.

Lemma 5.2. *Let $X_0 \sim X_1 \sim \cdots \sim X_k$ be a geodesic path of length $k \geqslant 1$ in Γ, so that $d(X_i, X_j) = |j - i|$, and let $\varepsilon, \varepsilon' \in \{\pm 1\}$. Then (X_0, ε) and (X_k, ε') have distance k or $k+1$ in $\widehat{\Gamma}$, according as $\sigma(X_0, X_k) = \varepsilon\varepsilon'$ or $-\varepsilon\varepsilon'$. In particular, the diameter of $\widehat{\Gamma}$ is* $\max\{n+1, 3\}$.

Proof. If $\sigma(X_0, X_k) = \varepsilon\varepsilon'$, then we have a path

$$(X_0, \varepsilon_0) \sim (X_1, \varepsilon_1) \sim \cdots \sim (X_k, \varepsilon_k)$$

in $\widehat{\Gamma}$ where $\varepsilon_i = \varepsilon_0\sigma(X_0, X_i)$ for $i = 1, 2, \ldots, k$; in particular if $\varepsilon_0 = \varepsilon$ then $\varepsilon_k = \varepsilon'$. Clearly this path in $\widehat{\Gamma}$ is shortest possible.

Now suppose $\sigma(X_0, X_k) = -\varepsilon\varepsilon'$. We first obtain a path

$$(X_0, \varepsilon) \sim (X_1, \varepsilon_1) \sim \cdots \sim (X_{k-1}, \varepsilon_{k-1})$$

in $\widehat{\Gamma}$ where $\varepsilon_i = \varepsilon_0\sigma(X_0, X_i)$ for $i = 1, 2, \ldots, k-1$. Let $Y \in \mathcal{V}$ be adjacent to both X_{k-1} and X_k in Γ, such that $\sigma(X_{k-1}, Y, X_k) = -1$. (By Lemma 4.6, there are $\frac{q-1}{2} \geqslant 1$ choices of such $Y \in \mathcal{V}$.) Appending

$$(X_{k-1}, \varepsilon_{k-1}) \sim (Y, \varepsilon'') \sim (X_k, \varepsilon'),$$

where $\varepsilon'' = \varepsilon_{k-1}\sigma(X_{k-1}, Y) = \varepsilon'\sigma(Y, X_k)$, we obtain a path of length $k+1$ from (X_0, ε) to (X_k, ε') in $\widehat{\Gamma}$; once again this path is shortest possible. $\square$

The fibers of the covering map $\widehat{\Gamma} \to \Gamma$ are the *antipodal* pairs $\{(X, 1), (X, -1)\}$ for $X \in \mathcal{V}$.

Lemma 5.3. *Let (X, ε) and (W, ε') be any two vertices of $\widehat{\Gamma}$. Then (X, ε) and (W, ε') are antipodal iff they are at distance 3 in $\widehat{\Gamma}$ and are joined by exactly $\frac{1}{2}q(q^n-1)$ paths of length 3 in $\widehat{\Gamma}$.*

Proof. Consider a typical antipodal pair $\{(X,1),(X,-1)\}$ where $X \in \mathcal{V}$. There exist $b_0 = q\left[{n \atop 1}\right]$ vertices $Y \in \mathcal{V}$ adjacent to X in Γ; and each such vertex Y has $a_1 = q-1$ neighbors Z in common with X. By Lemma 4.6, exactly half of these choices of the vertex Z yield coherent triples (X,Y,Z). In particular, X lies in exactly $b_0 \cdot \frac{1}{2}a_1 = \frac{1}{2}q\big(q^n-1\big)$ incoherent triples (X,Y,Z), giving the same number of paths $(X,1) \sim (Y,\varepsilon) \sim (Z,\varepsilon') \sim (X,-1)$ of length 3 in $\widehat{\Gamma}$. There is no path of length < 3 from $(X,1)$ to $(X,-1)$ in $\widehat{\Gamma}$, otherwise the covering map would give a closed path of length < 3 from X to X in Γ. This shows that any two antipodal vertices $(X,1),(X,-1)$ are at distance 3 in $\widehat{\Gamma}$; and in each case there are exactly $\frac{1}{2}q(q^n-1)$ geodesic paths from $(X,1)$ to $(X,-1)$.

Conversely, let (X,ε) and (W,ε') be any two vertices at distance 3 in $\widehat{\Gamma}$. By Lemma 5.2, $d(X,W) \in \{0,2,3\}$ in Γ. Consider first the case that $d(X,W) = 3$; then by Theorem 5.1, every geodesic path from (X,ε) to (W,ε') in $\widehat{\Gamma}$ arises from a unique geodesic path $X \sim Y \sim Z \sim W$ in Γ. There are exactly $c_3c_2c_1 = (q^2+q+1)(q+1)$ such geodesic paths from X to W; and this number clearly cannot equal $\frac{1}{2}q(q^n-1)$.

Next suppose $d(X,W) = 2$ in Γ. For every geodesic path

$$(X,\varepsilon) \sim (Y,\varepsilon'') \sim (Z,\varepsilon''') \sim (W,\varepsilon')$$

in $\widehat{\Gamma}$, we have $X \sim Y \sim Z \sim W$ in Γ. Further, the condition $d(X,W) = 2$ requires either $X \sim Z$ or $Y \sim W$ (but *not both,* by Lemma 4.2). We first count geodesic paths satisfying $X \sim Z$, noting that the vertex W has $c_2 = q+1$ neighbors Z in common with X; and in each case $\sigma(X,Z,W) = 1$ by Lemma 4.5. Moreover Z has $a_1 = q-1$ neighbors Y in common with X (all of which satisfy $\sigma(Y,Z,W) = 1$, again by Lemma 4.5). By the two-graph condition, we have $\sigma(X,Y,Z) = -1$ iff $\sigma(X,Y,W) = -1$. By Lemma 4.6, for each Z there are exactly $\frac{1}{2}\big(q-1\big)$ choices of Y satisfying the latter condition; and each such pair (Y,Z) yields a unique geodesic path $(X,\varepsilon) \sim (Y,\varepsilon'') \sim (Z,\varepsilon''') \sim (W,\varepsilon')$. We obtain $(q+1)\cdot\frac{1}{2}\big(q-1\big) = \frac{1}{2}\big(q^2-1\big)$ geodesic paths in this case. There are another $\frac{1}{2}\big(q^2-1\big)$ geodesic paths from (X,ε) to (W,ε') satisfying $Y \sim W$, for a total of q^2-1 geodesic paths. Once again, this number cannot equal $\frac{1}{2}q(q^n-1)$. □

Lemma 5.4. $\operatorname{Aut}\widehat{\Gamma}$ *acts naturally on* Γ*, with kernel* $\langle\zeta\rangle$*, inducing a* proper *subgroup* $\operatorname{Aut}\widehat{\Gamma}/\langle\zeta\rangle < \operatorname{Aut}\Gamma$.

Proof. By Lemma 5.3, $\operatorname{Aut}\widehat{\Gamma}$ permutes fibres of the covering map $\widehat{\Gamma} \to \Gamma$, and so $\operatorname{Aut}\widehat{\Gamma}$ acts naturally on Γ. It remains to be shown that the induced subgroup $\operatorname{Aut}\widehat{\Gamma}/\langle\zeta\rangle \leqslant \operatorname{Aut}\Gamma$ is proper.

Choose a hyperbolic basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ for V, so that $B(e_i,e_j) = B(f_i,f_j) = 0$ and $B(e_i,f_j) = \delta_{ij}$, and let $\eta \in \mathbb{F}_q$ be a nonsquare. Consider the subspaces $X,Y,Z,Z' \in \mathcal{V}$ defined by $X = \langle e_1,e_2,\ldots,e_n\rangle$, $Y = \langle f_1,e_2,\ldots,e_n\rangle$, $Z = \langle e_1+f_1,e_2,\ldots,e_n\rangle$ and $Z' = \langle e_1+\eta f_1,e_2,\ldots,e_n\rangle$. By straightforward computation, $\sigma(X,Y,Z) = 1$ and $\sigma(X,Y,Z') = -1$. Now consider $g \in GL(V)$ mapping our original ordered basis to the new ordered basis $e_1, \eta e_2, \ldots, \eta e_n, \eta f_1, f_2, \ldots, f_n$ so that $B(u^g,v^g) = \eta B(u,v)$ for all $u,v \in$

V; thus $g \in GSp(2n,q)$ induces an automorphism of the dual polar graph $\Gamma = \Gamma(2n,q)$. However, g maps the coherent triple $\{X,Y,Z\}$ to the non-coherent triple $\{X,Y,Z'\}$ and so does not preserve Δ_σ. If g were induced by an automorphism of $\widehat{\Gamma}$, this automorphism would map $\{X,Y,Z\} \times \{\pm 1\}$ to $\{X,Y,Z'\} \times \{\pm 1\}$. However, the induced subgraphs of $\widehat{\Gamma}$ on these two 6-sets of vertices are not isomorphic (a pair of triangles and a 6-cycle, respectively; see Section 3). $\square$

The natural action of $\Sigma Sp(2n,q)$ on $\mathcal{V}$ lifts to an action on $\widehat{\mathcal{V}}$ as follows: Let $g \in \Sigma Sp(2n,q)$ with associated field automorphism τ_g in the earlier notation of this section. Given $(U,\varepsilon) \in \widehat{\mathcal{V}}$, the map

$$U^n \to \mathbb{F}_q, \quad (u_1,u_2,\ldots,u_n) \mapsto \delta_{U^g}(u_1^g,u_2^g,\ldots,u_n^g)^{\tau_g^{-1}}$$

is a determinant function; so there exists a nonzero constant $\lambda_{g,U} \in \mathbb{F}_q$ such that

$$\delta_{U^g}(u_1^g,u_2^g,\ldots,u_n^g) = \lambda_{g,U}\,\delta_U(u_1,u_2,\ldots,u_n)^{\tau_g}.$$

Define $(U,\varepsilon)^g = (U^g, \chi(\lambda_{g,U})\varepsilon)$. One easily checks that this defines an action of $\Sigma Sp(2n,q)$ on $\widehat{\mathcal{V}}$. The central element $-I \in Sp(2n,q)$ fixes every $U \in \mathcal{V}$ and since

$$\delta_U(-u_1,-u_2,\ldots,-u_n) = (-1)^n \delta_U(u_1,u_2,\ldots,u_n)$$

where $\chi(-1)^n = 1$, $-I$ acts trivially on $\widehat{\mathcal{V}}$; thus $\Sigma Sp(2n,q)$ induces a permutation group $P\Sigma Sp(2n,q)$ on $\widehat{\mathcal{V}}$. The transposition ζ which exchanges antipodal vertices via $(U,1) \overset{\zeta}{\leftrightarrow} (U,-1)$ is not induced by any element of $P\Sigma Sp(2n,q)$ since $Z(P\Sigma Sp(2n,q)) = 1$, so we obtain a permutation group $\langle\zeta\rangle \times P\Sigma Sp(2n,q)$ acting faithfully on $\widehat{\mathcal{V}}$. We show that this permutation group preserves the graph $\widehat{\Gamma}$, and is in fact its full automorphism group:

Theorem 5.5. $\operatorname{Aut}\widehat{\Gamma} \cong 2 \times P\Sigma Sp(2n,q)$ *where this group acts as defined above. The full automorphism group of the two-graph associated to* σ *is* $\operatorname{Aut}\Delta_\sigma \cong P\Sigma Sp(2n,q)$.

Proof. Suppose $(X,\varepsilon) \sim (Y,\varepsilon')$ in $\widehat{\Gamma}$, so that $\sigma(X,Y) = \varepsilon\varepsilon'$; and let $g \in \Sigma Sp(2n,q)$ with $\tau_g \in \operatorname{Aut}\mathbb{F}_q$ as above. Then by Proposition 4.3(iv) we have

$$\sigma(X^g,Y^g) = \chi(\lambda_{g,X}\lambda_{g,Y})\sigma(X,Y) = \big(\chi(\lambda_{g,X})\varepsilon\big)\big(\chi(\lambda_{g,Y})\varepsilon'\big)$$

so that by definition, $(X,\varepsilon)^g \sim (Y,\varepsilon')^g$. Thus $P\Sigma Sp(2n,q)$, acting on $\widehat{\Gamma}$ as defined above, preserves the graph $\widehat{\Gamma}$. It is clear that the central factor $\langle\zeta\rangle$ also preserves $\widehat{\Gamma}$, so that $\operatorname{Aut}\widehat{\Gamma}$ has a subgroup isomorphic to $\langle\zeta\rangle \times P\Sigma Sp(2n,q)$. Moreover by Proposition 3.1, $\operatorname{Aut}\widehat{\Gamma}/\langle\zeta\rangle \cong \operatorname{Aut}\Delta_\sigma$. (We use the fact that by Lemma 5.4, $\operatorname{Aut}_\zeta\widehat{\Gamma} = \operatorname{Aut}\widehat{\Gamma}$ in the notation of Proposition 3.1.)

Suppose now that $n \geqslant 2$, so that $\operatorname{Aut}\Gamma \cong P\Gamma Sp(2n,q)$ by Theorem 4.1. By Lemma 5.4, $\operatorname{Aut}\widehat{\Gamma}$ acts on Γ, inducing a group of automorphisms satisfying

$$P\Sigma Sp(2n,q) \leqslant \operatorname{Aut}\widehat{\Gamma}/\langle\zeta\rangle < P\Gamma Sp(2n,q).$$

This forces $\operatorname{Aut}\widehat{\Gamma} \cong \langle\zeta\rangle \times P\Sigma Sp(2n,q)$ and $\operatorname{Aut}\Delta_\sigma \cong P\Sigma Sp(2n,q)$.

Finally suppose $n = 1$, so that Δ_σ is the Taylor-Paley two-graph on $q+1$ vertices, with full automorphism group $\operatorname{Aut}\Delta_\sigma \cong P\Sigma Sp(2,q) = P\Sigma L(2,q)$ by [18, Theorem 2]; see also [14, §4]. As above, $\operatorname{Aut}\widehat{\Gamma}$ has a subgroup isomorphic to $\langle\zeta\rangle \times P\Sigma Sp(2,q)$, and $\operatorname{Aut}\widehat{\Gamma}/\langle\zeta\rangle \cong \operatorname{Aut}\Delta_\sigma \cong P\Sigma Sp(2,q)$, so we must have equality: $\operatorname{Aut}\widehat{\Gamma} \cong 2 \times P\Sigma Sp(2,q) = 2 \times P\Sigma L(2,q)$. □

6. The Association Scheme

From the double cover $\widehat{\Gamma} \to \Gamma$, we now construct association schemes. As we will see in Section 7, this gives a new family of Q-polynomial association schemes. We begin with the relevant definitions, following [6, Chapter 2].

Let Ω be a finite set. A *(symmetric) d-class association scheme on* Ω is a pair $(\Omega, \mathcal{R})$ such that

1. $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $\Omega \times \Omega$;
2. R_0 is the identity relation on Ω;
3. $R_i = R_i^\top$ for $0 \leqslant i \leqslant d$; and
4. there are constants p_{ij}^k such that for any pair $(x,y) \in R_k$, the number of $z \in \Omega$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ equals p_{ij}^k.

For the rest of this paper, all association schemes are symmetric (i.e. the third property above holds). Each relation R_i has adjacency matrix A_i defined by

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } (x,y) \in R_i; \\ 0, & \text{otherwise.} \end{cases}$$

The axioms above imply that $p_{ji}^k = p_{ij}^k$ and the matrices $A_0, \ldots, A_d$ form an algebra of symmetric matrices satisfying $A_i A_j = \sum_k p_{ij}^k A_k$. This matrix algebra is also closed under Schur (entrywise) multiplication, which we will denote by '$\circ$'. This algebra is referred to as the *Bose-Mesner algebra* $\mathfrak{A}$ of the association scheme.

Since $\mathfrak{A}$ is a commutative algebra consisting of symmetric matrices, its elements are simultaneously diagonalizable, and $\mathfrak{A}$ has a second basis consisting of primitive idempotents $E_0, \ldots, E_d$. We define the parameters Q_{ij} by $E_j = \frac{1}{|\Omega|}\sum_i Q_{ij} A_i$. Similarly we define the parameters P_{ij} by the relation $A_j = \sum_i P_{ij} E_i$. The matrix P of parameters P_{ij} is often referred to as the character table of the scheme. The matrix Q of parameters Q_{ij} satisfies $Q = |\Omega| P^{-1}$.

We say an association scheme is *Q-polynomial* if, after suitably reindexing its idempotents, the idempotent E_j is a degree j polynomial in E_1 (where multiplication is done entrywise). This is equivalent to the condition that the jth column of the Q-matrix is a degree j polynomial of the column 1 of the Q-matrix (note that we start indexing the columns at 0).

Permutation groups give many examples of association schemes. Let G be a transitive permutation group acting on a finite set Ω, and suppose the orbits of G on $\Omega \times \Omega$ happen to be symmetric relations; such a group is called *generously*

transitive). It is not hard to check that the orbits of G on $\Omega \times \Omega$ form an association scheme. We will refer to these schemes as *Schurian schemes*.

We will now construct a $(2n{+}1)$-class Schurian association scheme $\mathcal{S} = \mathcal{S}_{n,q}$ with vertex set $\widehat{\mathcal{V}} = \mathcal{V} \times \{\pm 1\}$ of cardinality $|\widehat{\mathcal{V}}| = 2|\mathcal{V}| = 2q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$ using the Maslov index σ and the double cover $\widehat{\Gamma} \to \Gamma$ (defined as in Section 5). For $k = 0, 1, 2, \ldots, n$, the kth and $(2n{+}1{-}k)$th relations are given by

$$\begin{aligned} R_k &= \{((X,\varepsilon),(Y,\varepsilon')) \in \widehat{\mathcal{V}} \times \widehat{\mathcal{V}} \ :\ d(X,Y) = k,\ \varepsilon\varepsilon' = \sigma(X,Y)\}; \\ R_{2n+1-k} &= \{((X,\varepsilon),(Y,\varepsilon')) \in \widehat{\mathcal{V}} \times \widehat{\mathcal{V}} \ :\ d(X,Y) = k,\ \varepsilon\varepsilon' = -\sigma(X,Y)\}. \end{aligned}$$

These are symmetric relations which clearly partition $\widehat{\mathcal{V}} \times \widehat{\mathcal{V}}$. In particular, R_1 is the adjacency relation of our graph $\widehat{\Gamma}$ of Section 5; and the identity and antipodality relations are

$$\begin{aligned} R_0 &= \{((X,\varepsilon),(X,\varepsilon)) : X \in \mathcal{V},\ \varepsilon = \pm 1\}; \\ R_{2n+1} &= \{((X,\varepsilon),(X,-\varepsilon)) : X \in \mathcal{V},\ \varepsilon = \pm 1\}. \end{aligned}$$

We will write

$$(X,\varepsilon) \overset{i}{\sim} (Y,\varepsilon') \iff ((X,\varepsilon),(Y,\varepsilon')) \in R_i\,.$$

In the following, the parameters a_i, b_i, c_i are those of the dual polar graph Γ as given in Section 4.

Lemma 6.1. *Let $(X,\varepsilon) \overset{k}{\sim} (Y,\varepsilon')$ where $k \in \{0,1,2,\ldots,2n{+}1\}$. The number of $(Z,\varepsilon'') \in \widehat{\mathcal{V}}$ such that $(X,\varepsilon) \overset{i}{\sim} (Z,\varepsilon'') \overset{1}{\sim} (Y,\varepsilon')$ is*

$$p^k_{i,1} = \begin{cases} c_k = \left[{k \atop 1}\right], & \textit{if } i = k{-}1 \leqslant n; \\ \frac{1}{2}a_k = \frac{1}{2}\big(q^k - 1\big), & \textit{if } i = k; \\ b_k = q^{k+1}\left[{n-k \atop 1}\right], & \textit{if } i = k{+}1 \leqslant n{+}2; \\ b_{2n+1-k} = q^{2n+2-k}\left[{k-n-1 \atop 1}\right], & \textit{if } i = k{-}1 \geqslant n{-}1; \\ \frac{1}{2}a_{2n+1-k} = \frac{1}{2}\big(q^{2n+1-k} - 1\big), & \textit{if } i = 2n{+}1{-}k; \\ c_{2n+1-k} = \left[{2n+1-k \atop 1}\right], & \textit{if } i = k{+}1 \geqslant n{+}1; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. (i) First suppose $d(X,Y) = k \leqslant n$, so $\varepsilon\varepsilon' = \sigma(X,Y)$. Then $(Z,\varepsilon'') \in \widehat{\mathcal{V}}$ satisfies $(X,\varepsilon) \overset{i}{\sim} (Z,\varepsilon'') \overset{1}{\sim} (Y,\varepsilon')$ iff

$$\begin{cases} \qquad\text{case (i.a)} \\ \hline i = d(X,Z) \leqslant n \\ d(Z,Y) = 1 \\ \varepsilon'' = \varepsilon\sigma(X,Z) = \varepsilon'\sigma(Y,Z) \end{cases} \quad \textit{or} \quad \begin{cases} \qquad\text{case (i.b)} \\ \hline i = 2n{+}1{-}d(X,Z) \geqslant n{+}1 \\ d(Z,Y) = 1 \\ \varepsilon'' = -\varepsilon\sigma(X,Z) = \varepsilon'\sigma(Y,Z) \end{cases}$$

Moreover, each such (Z, ε'') satisfies $d(X, Z) \in \{k{-}1, k, k{+}1\}$ by the triangle inequality.

There are exactly $c_k = \left[{k \atop 1}\right]$ choices of $Z \in \mathcal{V}$ satisfying $d(X, Z) = k{-}1$ and $d(Z, Y) = 1$. Each such Z yields a coherent triple (X, Y, Z) by Lemma 4.5, so $\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)$. This yields c_k pairs (Z, ε''), all of which satisfy (i.a).

There are exactly $b_k = q^{k+1}\left[{n-k \atop 1}\right]$ choices of $Z \in \mathcal{V}$ satisfying $d(X, Z) = k{+}1$ and $d(Z, Y) = 1$. Each such Z yields a coherent triple (X, Y, Z) by Lemma 4.5, once again with $\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)$. This yields b_k pairs (Z, ε''), all of which satisfy (i.a).

There are exactly $a_k = q^k - 1$ choices of $Z \in \mathcal{V}$ satisfying $d(X, Z) = k$ and $d(Z, Y) = 1$. By Theorem 4.7, exactly $a_k/2$ of these Z yield coherent triples (X, Y, Z), in which case $\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)$; this yields $a_k/2$ pairs (Z, ε'') satisfying (i.a). The remaining $a_k/2$ of these Z yield incoherent triples (X, Y, Z), with $\varepsilon\varepsilon' = \sigma(X, Y) = -\sigma(X, Z)\sigma(Y, Z)$; and the resulting pairs (Z, ε'') satisfy (i.b). □

In Section 5 we lifted the action of $P\Sigma Sp(2n, q)$ on $\mathcal{V}$, to a transitive permutation action of $\langle\zeta\rangle \times P\Sigma Sp(2n, q)$ on $\widehat{\mathcal{V}}$ (below Lemma 5.4). Theorem 5.5 shows that this group preserves R_1 (the adjacency relation of the graph $\widehat{\Gamma}$). We next show that this group preserves *each* of the relations R_i, and so gives the full automorphism group of the scheme.

Lemma 6.2. *The diagonal action of* $2 \times PSp(2n, q)$ *on* $\widehat{\mathcal{V}} \times \widehat{\mathcal{V}}$ *preserves each of the relations* R_i*. The same conclusion holds for the subgroup* $2 \times P\Sigma Sp(2n, q)$*.*

Proof. Clearly the central factor $(U, \varepsilon) \overset{\zeta}{\leftrightarrow} (U, -\varepsilon)$ preserves each R_i. Now let $g \in Sp(2n, q)$, and suppose $X, Y \in \mathcal{V}$ such that $d(X, Y) = k \in \{0, 1, 2, \ldots, n\}$. Also let $\varepsilon, \varepsilon' \in \{\pm 1\}$, so that $((X, \varepsilon), (Y, \varepsilon')) \in R_k$ or R_{2n+1-k} according as $\varepsilon\varepsilon'\sigma(X, Y) = 1$ or -1. Since g preserves distances in Γ, $d(X^g, Y^g) = k$. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be bases for X and Y respectively, such that a basis for $X \cap Y$ is formed by $x_{k+1}{=}y_{k+1}$, $x_{k+2}{=}y_{k+2}$, $\ldots$, $x_n{=}y_n$. Then $(X, \varepsilon)^g = (X^g, \chi(\lambda_{g,X})\varepsilon)$ and $(Y, \varepsilon')^g = (Y^g, \chi(\lambda_{g,Y})\varepsilon')$ where

$$\begin{aligned}
&\chi(\lambda_{g,X})\varepsilon\chi(\lambda_{g,Y})\varepsilon'\sigma(X^g, Y^g) \\
&\qquad = \varepsilon\varepsilon'\chi\big(\lambda_{g,X}\delta_{X^g}(x_1^g, \ldots, x_n^g)\lambda_{g,Y}\delta_{Y^g}(y_1^g, \ldots, y_n^g) \\
&\qquad\quad \times \det\big[B(x_i^g, y_j^g) : 1 \leqslant i, j \leqslant k\big]\big) \\
&\qquad = \varepsilon\varepsilon'\chi\big(\lambda_{g,X}^2\lambda_{g,Y}^2\delta_X(x_1, \ldots, x_n)\delta_Y(y_1, \ldots, y_n)\det\big[B(x_i, y_j) \,:\, 1 \leqslant i, j \leqslant k\big]\big) \\
&\qquad = \varepsilon\varepsilon'\sigma(X, Y).
\end{aligned}$$

If this value is 1, then both $(X, \varepsilon) \overset{k}{\sim} (Y, \varepsilon')$ and $(X, \varepsilon)^g \overset{k}{\sim} (Y, \varepsilon')^g$; but if the latter value is -1, then $(X, \varepsilon) \overset{2n+1-k}{\sim} (Y, \varepsilon')$ and $(X, \varepsilon)^g \overset{2n+1-k}{\sim} (Y, \varepsilon')^g$.

Thus $2 \times PSp(2n, q)$ preserves the relations R_i as claimed. A similar argument holds for $2 \times P\Sigma Sp(2n, q)$. □

It is easy to see that $\langle\zeta\rangle \times P\Sigma Sp(2n,q)$ acts transitively on each R_i, and similarly for $\langle\zeta\rangle \times PSp(2n,q)$. This yields

Theorem 6.3. *The diagonal action of the group $2\times PSp(2n,q)$ on $\widehat{\mathcal{V}}\times\widehat{\mathcal{V}}$ has orbits R_0, R_1, …, R_{2n+1}; so these form the relations of a $(2n+1)$-class Schurian association scheme. The same conclusion holds for $2\times P\Sigma Sp(2n,q)$, which is therefore the full automorphism group of the association scheme $\mathcal{S}$.* □

7. The Q-polynomial property

In this section we will use some parameters of the scheme to prove that the association scheme $\mathcal{S}$ is Q-polynomial. We will benefit from the action of the A_i's by left-multiplication on the Bose-Mesner algebra, resulting in matrices L_i defined by $(L_i)_{kj} = p^k_{ij}$. In particular, the parameter p^k_{1j} of the scheme from Lemma 6.1, is the (k,j)-entry of the matrix

$$L_1 = \begin{pmatrix}
0 & b_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
1 & \frac{a_1}{2} & b_1 & 0 & 0 & \cdots & 0 & 0 & \frac{a_1}{2} & 0\\
0 & c_2 & \frac{a_2}{2} & \ddots & 0 & \cdots & 0 & \iddots & 0 & 0\\
0 & 0 & \ddots & \ddots & b_{d-1} & 0 & \frac{a_{d-1}}{2} & 0 & 0 & 0\\
0 & 0 & 0 & c_d & \frac{a_d}{2} & \frac{a_d}{2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{a_d}{2} & \frac{a_d}{2} & c_d & 0 & 0 & 0\\
0 & 0 & 0 & \frac{a_{d-1}}{2} & 0 & b_{d-1} & \ddots & \ddots & 0 & 0\\
0 & 0 & \iddots & 0 & 0 & 0 & \ddots & \frac{a_2}{2} & c_2 & 0\\
0 & \frac{a_1}{2} & 0 & 0 & 0 & 0 & 0 & b_1 & \frac{a_1}{2} & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0
\end{pmatrix}.$$

As it turns out, this matrix has distinct eigenvalues, which in turn will give us a great deal of information about the scheme. In particular by [6, Proposition 2.2.2], the columns of Q are right eigenvectors of L_1. We will use the following generalization of [6, Theorem 8.1.1] to prove that $\mathcal{S}$ is Q-polynomial.

Theorem 7.1. *Suppose A_i is a matrix in a d-class association scheme $(\Omega,\mathcal{R})$ with $d+1$ distinct eigenvalues. Then $(\Omega,\mathcal{R})$ is Q-polynomial if and only if there is a sequence of* distinct *complex numbers $\sigma_0,\sigma_1,\ldots,\sigma_d$ and polynomials $s_0(x), s_1(x),\ldots,s_d(x)$ of degree $0,1,\ldots,d$, respectively, with*

$$\sum_j p^k_{ij}\sigma^\ell_j = s_\ell(\sigma_k)$$

for $0 \leqslant \ell \leqslant d$. Furthermore, the leading coefficients of the polynomials $s_0(x)$, $s_1(x)$, …, $s_d(x)$ are precisely the eigenvalues of A_i in a Q-polynomial ordering.

Proof. Without loss of generality we assume A_1 has this property. Let L_1 be the corresponding intersection matrix. Let S, T be the $d{+}1$ by $d{+}1$ matrices with

$S_{jk} = \sigma_j^k$ and T_{jk} equal to the coefficient of x^j in the polynomial $s_k(x)$. Then the above statement is equivalent to $L_1 S = ST$. Then L_i is similar to T and since T is upper triangular, the diagonal entries of T are precisely the eigenvalues of L_1. Since T is an upper triangular matrix with distinct diagonal entries, an easy induction shows that it can be diagonalized by an upper triangular matrix. Namely, there is an invertible matrix U and a diagonal matrix D with $D_{jj} = T_{jj}$ such that $U^{-1}TU = D$. Then $L_1(SU) = (SU)D$. This implies that the columns of SU are eigenvectors of L_1, hence there is a diagonal matrix D' such that $SUD' = Q$. Since the jth column of SUD' is a degree j polynomial of the first column of T, which is a linear combination of columns 0 and 1 of SUD', it is clear that the jth column of SUD' is a degree j polynomial of the first column of SUD'. This implies that the columns of Q are in a given Q-polynomial ordering, which in turn implies that the ordering of the eigenvalues in T is a Q-polynomial ordering. □

This leads to our main result:

Theorem 7.2. *The scheme $\mathcal{S}$ is Q-polynomial. Furthermore, it has two Q-polynomial orderings.*

Proof. Let $r = \sqrt{q}$ and $d = 2n+1$. We define the sequence of polynomials

$$s_\ell(x) = \begin{cases} r^\ell \begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} x^\ell + \frac{1}{r^{\ell-2}} \begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} x^{\ell-2}, & \text{for } \ell \text{ odd;} \\ r^\ell \left(\begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} - \frac{1}{r^\ell} \right) x^\ell + \frac{1}{r^{\ell-2}} \left(\begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} + r^{\ell-2} \right) x^{\ell-2}, & \text{for } \ell \text{ even} \end{cases}$$

and constants

$$\sigma_j = \begin{cases} \frac{1}{r^j}, & \text{for } 0 \leqslant j \leqslant n; \\ -\frac{1}{r^{2n+1-j}}, & \text{for } n+1 \leqslant j \leqslant 2n+1. \end{cases}$$

The polynomials $s_0(x), \ldots, s_{2n+1}(x)$ realize $\sigma_0, \ldots, \sigma_{2n+1}$ as a Q-sequence for $\mathcal{S}$, as we proceed to show by direct computation. For $k \leqslant n$ we have $\sum_j p_{1j}^k \sigma_j^\ell = c_k \sigma_{k-1}^\ell + \frac{a_k}{2} \sigma_k^\ell + b_k \sigma_{k+1}^\ell + \frac{a_k}{2} \sigma_{2n+1-k}^\ell$. For odd ℓ this reduces to

$$\begin{aligned} c_k \sigma_{k-1}^\ell + b_k \sigma_{k+1}^\ell &= \frac{1}{r^{(k-1)\ell}} \begin{bmatrix} k \\ 1 \end{bmatrix} + q \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix} \right) \frac{1}{r^{(k+1)\ell}} \\ &= \frac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} q^\ell + q \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix} \right) \right) \\ &= \frac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} n+1 \\ 1 \end{bmatrix} - \begin{bmatrix} \ell \\ 1 \end{bmatrix} + \begin{bmatrix} k+\ell \\ 1 \end{bmatrix} - \begin{bmatrix} k+1 \\ 1 \end{bmatrix} \right) \\ &= \frac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} r^{2\ell} + \begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} r^{2(k+1)} \right) \\ &= r^\ell \begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} \frac{1}{r^{k\ell}} + \frac{1}{r^{\ell-2}} \begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} \frac{1}{r^{k(\ell-2)}} \\ &= s_\ell(\sigma_k), \end{aligned}$$

whereas for even ℓ we have

$$\textstyle\sum_j p_{1j}^k \sigma_j^\ell = c_k \sigma_{k-1}^\ell + a_k \sigma_k^\ell + b_k \sigma_{k+1}^\ell$$

$$
\begin{aligned}
&= \begin{bmatrix} k \\ 1 \end{bmatrix} \tfrac{1}{r^{(k-1)\ell}} + (q-1) \begin{bmatrix} k \\ 1 \end{bmatrix} \tfrac{1}{r^{k\ell}} + q \left(\begin{bmatrix} k \\ 1 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix} \right) \tfrac{1}{r^{(k+1)\ell}} \\
&= \tfrac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} k \\ 1 \end{bmatrix} q^\ell + q\big(\begin{bmatrix} n \\ 1 \end{bmatrix} - \begin{bmatrix} k \\ 1 \end{bmatrix} \big) + r^{2k+\ell} - r^\ell \right) \\
&= \tfrac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} n+1 \\ 1 \end{bmatrix} - \begin{bmatrix} \ell \\ 1 \end{bmatrix} + \begin{bmatrix} k+\ell \\ 1 \end{bmatrix} - \begin{bmatrix} k+1 \\ 1 \end{bmatrix} + r^{2k+\ell} - r^\ell \right) \\
&= \tfrac{1}{r^{(k+1)\ell}} \left(\begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} r^{2\ell} - r^\ell + \begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} r^{2(k+1)} + r^{2k+\ell} \right) \\
&= r^\ell \left(\begin{bmatrix} n-\ell+1 \\ 1 \end{bmatrix} - \tfrac{1}{r^\ell} \right) \tfrac{1}{r^{k\ell}} + \tfrac{1}{r^{\ell-2}} \left(\begin{bmatrix} \ell-1 \\ 1 \end{bmatrix} + r^{\ell-2} \right) \tfrac{1}{r^{k(\ell-2)}} \\
&= s_\ell(\sigma_k).
\end{aligned}
$$

Now we deal with $k \geqslant n+1$, noting that

$$\sum_j p^k_{1j} \sigma^\ell_j = b_{2n+1-k} \sigma^\ell_{k-1} + \tfrac{a_{2n+1-k}}{2} \sigma^\ell_k + c_{2n+1-k} \sigma^\ell_{k+1} + \tfrac{a_{2n+1-k}}{2} \sigma^\ell_{2n+1-k}.$$

For odd ℓ this reduces to

$$
\begin{aligned}
b_{2n+1-k} \sigma^\ell_{k-1} + c_{2n+1-k} \sigma^\ell_{k+1} &= -b_{2n+1-k} \sigma^\ell_{2n+2-k} - c_{2n+1-k} \sigma^\ell_{2n-k} \\
&= -s_\ell(\sigma_{2n+1-k}) = s_\ell(-\sigma_{2n+1-k}) = s_\ell(\sigma_k),
\end{aligned}
$$

while for even ℓ we obtain

$$
\begin{aligned}
&b_{2n+1-k} \sigma^\ell_{k-1} + a_{2n+1-k} \sigma^\ell_k + c_{2n+1-k} \sigma^\ell_{k+1} \\
&\qquad = b_{2n+1-k} \sigma^\ell_{2n+2-k} + a_{2n+1-k} \sigma^\ell_{2n+1-k} + c_{2n+1-k} \sigma^\ell_{2n-k} \\
&\qquad = s_\ell(\sigma_{2n+1-k}) = s_\ell(-\sigma_{2n+1-k}) = s_\ell(\sigma_k).
\end{aligned}
$$

For nonsquare q the splitting field of $\mathcal{S}$ is irrational, implying that it is a quadratic extension of the rationals, namely $\mathbb{Q}(r)$. The Galois group acts faithfully on the idempotents of the scheme, yielding a second Q-polynomial ordering. This second Q-polynomial ordering can also be obtained by replacing $r \mapsto -r$ in both the σ_j and the polynomials $s_\ell(x)$, showing that this second ordering exists for square q as well. $\square$

We note that by a result of Suzuki [17], Q-polynomial schemes can have at most two Q-polynomial orderings.

8. The P-matrix

We now compute the P-matrix of the scheme $\mathcal{S}$, expressing it in terms of the auxiliary matrices $\widetilde{P}$ and $\widehat{P}$ whose entries are defined by

$$
\begin{aligned}
\widetilde{P}_{ij} &= \sum_{l=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix}; \\
\widehat{P}_{ij} &= \sum_{\ell=0}^{j} (-1)^\ell r^{(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix}.
\end{aligned}
$$

By [6, Proposition 2.2.2], the P-matrix is determined by the left-normalized left eigenvectors of L_1. We first show that the rows of $\widetilde{P}$ and $\widehat{P}$ are left eigenvectors of the matrices defined by

$$\widetilde{M} = \begin{pmatrix} 0 & b_0 & & & & \\ 1 & a_1 & b_1 & & & \\ & c_2 & a_2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{pmatrix}, \quad \widehat{M} = \begin{pmatrix} 0 & b_0 & & & & \\ 1 & 0 & b_1 & & & \\ & c_2 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & b_{d-1} \\ & & & & c_d & 0 \end{pmatrix}$$

respectively. We will show that the corresponding diagonal forms are

$$\widetilde{D} = \operatorname{diag}(\widetilde{P}_{i0}, \widetilde{P}_{i1}, \dots, \widetilde{P}_{in}), \quad \widehat{D} = \operatorname{diag}(\widehat{P}_{i0}, \widehat{P}_{i1}, \dots, \widehat{P}_{in}).$$

The ordering we give to the eigenvectors of $\widetilde{M}$ and $\widehat{M}$ may seem arbitrary, but will be important later.

Theorem 8.1. *$\widetilde{P}\widetilde{M} = \widetilde{D}\widetilde{P}$ and $\widehat{P}\widehat{M} = \widehat{D}\widehat{P}$.*

Proof. Fix i and let $v_i = (\widetilde{P}_{i0}, \widetilde{P}_{i1}, \dots, \widetilde{P}_{in})$. We must show that $v_i\widetilde{M} = \widetilde{P}_{i1}v_i$. In particular, we need to show the following recurrence holds for all j:

$$\begin{aligned} b_{j-1}\sum_{\ell=0}^{j-1}(-1)^\ell r^{j-1-2\ell+(j-1-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-1-\ell \end{bmatrix} + a_j\sum_{\ell=0}^{j}(-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell \end{bmatrix} \\ + c_{j+1}\sum_{\ell=0}^{j+1}(-1)^\ell r^{j+1-2\ell+(j+1-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j+1-\ell \end{bmatrix} \\ = \widetilde{P}_{i1}\sum_{\ell=0}^{j}(-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell \end{bmatrix}. \end{aligned}$$

Multiplying both sides by $q-1$ and substituting for b_{j-1}, a_j and c_{j+1}, we find this is equivalent to showing that the quantity z_j, defined as follows, vanishes for all j:

$$\begin{aligned} z_j = (q^{n+1} - q^j)\sum_{\ell=0}^{j-1}(-1)^\ell r^{j-2\ell-1+(j-\ell-1)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} \\ + (q-1)(q^j-1)\sum_{\ell=0}^{j}(-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell \end{bmatrix} \\ + (q^{j+1}-1)\sum_{\ell=0}^{j+1}(-1)^\ell r^{j-2\ell+1+(j-\ell+1)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} \\ - \big(q^i(q^{n-2i+1}-1) - q + 1\big)\sum_{\ell=0}^{j}(-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2}\begin{bmatrix} i \\ \ell \end{bmatrix}\begin{bmatrix} n-i \\ j-\ell \end{bmatrix}. \end{aligned}$$

The second and last sums combine, simplifying to

$$\begin{aligned}
z_j = (q^{n+1} - q^j) \sum_{\ell=0}^{j-1} (-1)^\ell r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} \\
+ \left((q-1)q^j + q^i - q^{n-i+1}\right) \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix} \\
+ (q^{j+1} - 1) \sum_{\ell=0}^{j+1} (-1)^\ell r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix}.
\end{aligned}$$

Now it suffices to show that the generating function $Z(t) = \sum_{j=0}^\infty z_j t^j$ vanishes. We first express $Z(t)$ in terms of the polynomials $E_m(t)$ defined in Section 2. Using Proposition 2.2(iii), we are able to rewrite our generating function as $Z(t) = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_6$ where

$$\begin{aligned}
\Sigma_1 &= q^{n+1} \sum_{j=0}^\infty \sum_{\ell=0}^{j-1} (-1)^\ell r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} t^j \\
&= q^{n+1} t E_i(-t) E_{n-i}(qt); \\
\Sigma_2 &= -\sum_{j=0}^\infty q^j \sum_{\ell=0}^{j-1} (-1)^\ell r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} t^j \\
&= -qt E_i(-qt) E_{n-i}(q^2 t); \\
\Sigma_3 &= (q-1) \sum_{j=0}^\infty q^j \sum_{\ell=0}^{j+1} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix} t^j \\
&= (q-1) E_i(-qt) E_{n-i}(q^2 t); \\
\Sigma_4 &= (q^i - q^{n-i+1}) \sum_{j=0}^\infty \sum_{\ell=0}^{j+1} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix} t^j \\
&= (q^i - q^{n-i+1}) E_i(-t) E_{n-i}(qt); \\
\Sigma_5 &= \sum_{j=0}^\infty q^{j+1} \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} t^j \\
&= \tfrac{1}{t} E_i(-qt) E_{n-i}(q^2 t); \\
\Sigma_6 &= -\sum_{j=0}^\infty \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} t^j \\
&= -\tfrac{1}{t} E_i(-t) E_{n-i}(qt).
\end{aligned}$$

Using Proposition 2.2(i,ii), we find

$$\begin{aligned} Z(t) &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 \\ &= \Big(q^{n+1} t + q^i - q^{n-i+1} - \tfrac{1}{t} \\ &\qquad + \tfrac{(1-q^i t)}{(1-t)} \tfrac{(1+q^{n-i+1} t)}{(1+qt)} \big(-qt + q - 1 + \tfrac{1}{t} \big) \Big) E_i(-t) E_{n-i}(qt) = 0 \end{aligned}$$

as required.

The strategy for showing $\widehat{P}\widehat{M} = \widehat{D}\widehat{P}$ is very similar but the details are sufficiently different that we provide the details here. Fix i and let $v_i = (\widehat{P}_{i0}, \widehat{P}_{i1}, \ldots, \widehat{P}_{in})$. We must show that $v_i \widehat{M} = \widehat{P}_{i1} v_i$. In particular, we need to show the following recurrence holds for all j:

$$\begin{aligned} b_{j-1} \sum_{\ell=0}^{j-1} (-1)^\ell r^{(j-1-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-1-\ell \end{bmatrix} + c_{j+1} \sum_{\ell=0}^{j+1} (-1)^\ell r^{(j+1-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j+1-\ell \end{bmatrix} \\ = \widehat{P}_{i1} \sum_{\ell=0}^{j} (-1)^\ell r^{(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix}. \end{aligned}$$

Multiplying both sides by $q-1$ and substituting for b_{j-1}, c_{j+1}, we find this is equivalent to showing that the following is zero for all j:

$$\begin{aligned} z_j = (q^{n+1} - q^j) &\sum_{\ell=0}^{j-1} (-1)^\ell r^{(j-\ell-1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} \\ + (q^{j+1} - 1) &\sum_{\ell=0}^{j+1} (-1)^\ell r^{(j-\ell+1)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} \\ - q^i (q^{2n-i} - 1) &\sum_{\ell=0}^{j} (-1)^\ell r^{(j-\ell)^2+\ell^2} \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell \end{bmatrix}. \end{aligned}$$

Again, it suffices to show that the generating function $Z(t) = \sum_{j=0}^{\infty} z_j t^j$ vanishes. As before, we first rewrite our generating function as $Z(t) = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_6$ where

$$\begin{aligned} \Sigma_1 &= q^{n+1} \sum_{\ell=0}^{j-1} (-1)^\ell r^{(j-\ell-1)^2+\ell^2} t^\ell \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} t^m = q^{n+1} t E_i(-rt) E_{n-i}(rt); \\ \Sigma_2 &= -q^j \sum_{\ell=0}^{j-1} (-1)^\ell r^{(j-\ell-1)^2+\ell^2} t^\ell \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell-1 \end{bmatrix} t^m = -qt E_i(-r^3 t) E_{n-i}(r^3 t); \\ \Sigma_3 &= q^{j+1} \sum_{\ell=0}^{j+1} (-1)^\ell r^{(j-\ell+1)^2+\ell^2} t^\ell \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} t^m = \tfrac{1}{t} E_i(-r^3 t) E_{n-i}(r^3 t); \\ \Sigma_4 &= -\sum_{\ell=0}^{j+1} (-1)^\ell r^{(j-\ell+1)^2+\ell^2} t^\ell \begin{bmatrix} i \\ \ell \end{bmatrix} \begin{bmatrix} n-i \\ j-\ell+1 \end{bmatrix} t^m = -\tfrac{1}{t} E_i(-rt) E_{n-i}(rt); \end{aligned}$$

$$\Sigma_5 = -rq^{2n}\sum_{\ell=0}^{j}(-1)^\ell r^{(j-\ell)^2+\ell^2} t^\ell \left[\begin{smallmatrix} i \\ \ell \end{smallmatrix}\right]\left[\begin{smallmatrix} n-i \\ j-\ell \end{smallmatrix}\right] t^m = -rq^{n-i}E_i(-rt)E_{n-i}(rt);$$

$$\Sigma_6 = rq^i\sum_{\ell=0}^{j}(-1)^\ell r^{(j-\ell)^2+\ell^2} t^\ell \left[\begin{smallmatrix} i \\ \ell \end{smallmatrix}\right]\left[\begin{smallmatrix} n-i \\ j-\ell \end{smallmatrix}\right] t^m = rq^iE_i(-rt)E_{n-i}(rt)$$

in terms of the polynomials $E_m(t)$ defined in Section 2. Using Proposition 2.2(iii), we find

$$\begin{aligned} Z(t) &= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6 \\ &= \Big(q^{n+1}t - \tfrac{1}{t} - rq^{n-i} + rq^i \\ &\qquad + \left(\tfrac{1}{t} - qt\right)\tfrac{(1-rq^it)(1+rq^{n-i}t)}{(1-rt)(1+rt)}\Big)E_i(-rt)E_{n-i}(rt) = 0 \end{aligned}$$

as required. □

Corollary 8.2. *The P-matrix of the Q-polynomial scheme $\mathcal{S}$ is given by*

$$\begin{cases} P_{i,j} = P_{i,2n+1-j} = \widetilde{P}_{\frac{i}{2},j} & \text{for } i \text{ even, } 0 \leqslant j \leqslant n; \\ P_{i,j} = -P_{i,2n+1-j} = \widehat{P}_{\lfloor\frac{i}{2}\rfloor,j} & \text{for } i \text{ odd, } 0 \leqslant j \leqslant n. \end{cases}$$

Proof. If $(v_0,\dots,v_n)$ is a left eigenvector of $\widetilde{M}$ or $\widehat{M}$, it is easily seen that either $(v_0,\dots,v_n,v_n,\dots,v_0)$ or $(v_0,\dots,v_n,-v_n,\dots,-v_0)$ is a left eigenvector of L_1, respectively. The fact that this ordering of the eigenvalues of L_1 is a Q-polynomial ordering follows from Theorem 7.2. □

9. A hypothetical subscheme

We ask whether $\mathcal{S}$ is the extended Q-bipartite double (in the sense of [13]) of a primitive Q-polynomial scheme. We investigated these parameters up to $n = 20$ and found they satisfied the Krein conditions, had integral eigenvalue multiplicities and nonnegative integral p_{ij}^k, and satisfy the handshaking lemma for all square q. This appears to give an infinite family of feasible parameters for primitive Q-polynomial schemes with an unbounded number of classes. Detailed parameters and a proof of feasibility will be given in a forthcoming paper of Eiichi Bannai and Jianmin Ma.

We give the smallest case below for which existence is unknown:

$$P = \begin{pmatrix} 1 & \frac{r^4+r^3+r^2+r}{2} & \frac{r^4-r^3+r^2-r}{2} & \frac{r^6+r^4}{2} & \frac{r^6-r^4}{2} \\ 1 & \frac{r^3+r^2+r-1}{2} & \frac{-r^3+r^2-r-1}{2} & \frac{r^4-r^2}{2} & \frac{-r^4-r^2}{2} \\ 1 & \frac{r^2-1}{2} & \frac{r^2-1}{2} & -r^2 & 0 \\ 1 & \frac{-r^2-1}{2} & \frac{-r^2-1}{2} & 0 & r^2 \\ 1 & \frac{-r^3-r^2-r-1}{2} & \frac{r^3-r^2+r-1}{2} & \frac{r^4+r^2}{2} & \frac{-r^4+r^2}{2} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & \frac{r^4-1}{2} & \frac{r^6+r^4+r^2+1}{2} & \frac{r^6-r^4+r^2-1}{2} & \frac{r^4+1}{2} \\ 1 & \frac{r^4-2r+1}{2r} & \frac{r^5-r^4+r-1}{2r} & \frac{-r^5+r^4-r+1}{2r} & \frac{-r^4-1}{2r} \\ 1 & \frac{-r^4-2r-1}{2r} & \frac{r^5+r^4+r+1}{2r} & \frac{-r^5-r^4-r-1}{2r} & \frac{r^4+1}{2r} \\ 1 & \frac{r^4-2r^2+1}{2r^2} & \frac{-r^4-1}{r^2} & 0 & \frac{r^4+1}{2r^2} \\ 1 & \frac{-r^4-2r^2-1}{2r^2} & 0 & \frac{r^4+1}{r^2} & \frac{-r^4-1}{2r^2} \end{pmatrix}$$

$$L_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 0 & \frac{r^4+r^3+r^2+r}{2} & 0 & 0 & 0 \\ 1 & \frac{r^2+2r-3}{4} & \frac{r^2-1}{4} & \frac{r^4+r^3}{2} & 0 \\ 0 & \frac{r^2+2r+1}{4} & \frac{r^2-1}{4} & 0 & \frac{r^4+r^3}{2} \\ 0 & \frac{r^2+2r+1}{2} & 0 & \frac{r^4+r^3+r^2-r-2}{4} & \frac{r^4+r^3-r^2-r}{4} \\ 0 & 0 & \frac{r^2+1}{2} & \frac{r^4+r^3+r^2+r}{4} & \frac{r^4+r^3-r^2+r-2}{4} \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 0 & \frac{r^4-r^3+r^2-r}{2} & 0 & 0 \\ 0 & \frac{r^2-1}{4} & \frac{r^2-2r+1}{4} & 0 & \frac{r^4-r^3}{2} \\ 1 & \frac{r^2-1}{4} & \frac{r^2-2r-3}{4} & \frac{r^4-r^3}{2} & 0 \\ 0 & 0 & \frac{r^2-2r+1}{2} & \frac{r^4-r^3+r^2+r-2}{4} & \frac{r^4-r^3-r^2+r}{4} \\ 0 & \frac{r^2+1}{2} & 0 & \frac{r^4-r^3+r^2-r}{4} & \frac{r^4-r^3-r^2-r-2}{4} \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & \frac{r^6+r^4}{2} & 0 \\ 0 & \frac{r^4+r^3}{2} & 0 & \frac{r^6+r^4-2r^3}{4} & \frac{r^6-r^4}{4} \\ 0 & 0 & \frac{r^4-r^3}{2} & \frac{r^6+r^4+2r^3}{4} & \frac{r^6-r^4}{4} \\ 1 & \frac{r^4+r^3+r^2-r-2}{4} & \frac{r^4-r^3+r^2+r-2}{4} & \frac{r^6+2r^4-3r^2}{4} & \frac{r^6-2r^4+r^2}{4} \\ 0 & \frac{r^4+r^3+r^2+r}{4} & \frac{r^4-r^3+r^2-r}{4} & \frac{r^6-r^2}{4} & \frac{r^6-r^2}{4} \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{r^6-r^4}{2} \\ 0 & 0 & \frac{r^4-r^3}{2} & \frac{r^6-r^4}{4} & \frac{r^6-3r^4+2r^3}{4} \\ 0 & \frac{r^4+r^3}{2} & 0 & \frac{r^6-r^4}{4} & \frac{r^6-3r^4-2r^3}{4} \\ 0 & \frac{r^4+r^3-r^2-r}{4} & \frac{r^4-r^3-r^2+r}{4} & \frac{r^6-2r^4+r^2}{4} & \frac{r^6-2r^4+r^2}{4} \\ 1 & \frac{r^4+r^3-r^2+r-2}{4} & \frac{r^4-r^3-r^2-r-2}{4} & \frac{r^6-r^2}{4} & \frac{r^6-4r^4+3r^2}{4} \end{pmatrix}$$

$$L_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L_1^* = \begin{pmatrix} 0 & \frac{r^4-1}{2} & 0 & 0 & 0 \\ 1 & \frac{r^6-5r^4-3r^2-1}{2(r^4+r^2)} & \frac{r^8+2r^4+1}{2(r^4+r^2)} & 0 & 0 \\ 0 & \frac{r^6-r^4+r^2-1}{2(r^4+r^2)} & \frac{r^8-4r^2+3}{4(r^4+r^2)} & \frac{r^6-r^4+r^2-1}{4r^2} & 0 \\ 0 & 0 & \frac{r^6+r^4+r^2+1}{4r^2} & \frac{r^6-3r^4-3r^2-3}{4r^2} & \frac{r^4+1}{2r^2} \\ 0 & 0 & 0 & \frac{r^6-r^4+r^2-1}{2r^2} & \frac{r^4-2r^2+1}{2r^2} \end{pmatrix}$$

$$L_2^* = \begin{pmatrix} 0 & 0 & \frac{r^6+r^4+r^2+1}{2} & 0 & 0 \\ 0 & \frac{r^8+2r^4+1}{2(r^4+r^2)} & \frac{r^{10}+r^8+2r^6-2r^4+r^2-3}{4(r^4+r^2)} & \frac{r^8+2r^4+1}{4r^2} & 0 \\ 1 & \frac{r^8-4r^2+3}{4(r^4+r^2)} & \frac{r^{10}+3r^8+2r^6-2r^4+r^2-5}{4(r^4+r^2)} & \frac{r^8-2r^6+2r^4-2r^2+1}{4r^2} & \frac{r^6+r^4+r^2+1}{4r^2} \\ 0 & \frac{r^6+r^4+r^2+1}{4r^2} & \frac{r^8-1}{4r^2} & \frac{r^8+2r^4+1}{4r^2} & \frac{r^6-r^4+r^2-1}{4r^2} \\ 0 & 0 & \frac{r^8+2r^6+2r^4+2r^2+1}{4r^2} & \frac{r^8-2r^6+2r^4-2r^2+1}{4r^2} & \frac{r^6-r^4+r^2-1}{2r^2} \end{pmatrix}$$

$$L_3^* = \begin{pmatrix} 0 & 0 & 0 & \frac{r^6-r^4+r^2-1}{2} & 0 \\ 0 & 0 & \frac{r^8+2r^4+1}{4r^2} & \frac{r^{10}-3r^8-2r^6-6r^4-3r^2-3}{4(r^4+r^2)} & \frac{r^8+2r^4+1}{2(r^4+r^2)} \\ 0 & \frac{r^6-r^4+r^2-1}{4r^2} & \frac{r^8-2r^6+2r^4-2r^2+1}{4r^2} & \frac{r^{10}-r^8+2r^6-2r^4+r^2-1}{4(r^4+r^2)} & \frac{r^8-2r^6+2r^4-2r^2+1}{4(r^4+r^2)} \\ 1 & \frac{r^6-3r^4-3r^2-3}{4r^2} & \frac{r^8+2r^4+1}{4r^2} & \frac{r^8-4r^6+4r^4-4r^2+3}{4r^2} & \frac{r^6-r^4+r^2-1}{4r^2} \\ 0 & \frac{r^6-r^4+r^2-1}{2r^2} & \frac{r^8-2r^6+2r^4-2r^2+1}{4r^2} & \frac{r^8-2r^6+2r^4-2r^2+1}{4r^2} & 0 \end{pmatrix}$$

$$L_4^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{r^4+1}{2} \\ 0 & 0 & 0 & \frac{r^8+2r^4+1}{2(r^4+r^2)} & \frac{r^6-r^4+r^2-1}{2(r^4+r^2)} \\ 0 & 0 & \frac{r^6+r^4+r^2+1}{4r^2} & \frac{r^8-2r^6+2r^4-2r^2+1}{4(r^4+r^2)} & \frac{r^6-r^4+r^2-1}{2(r^4+r^2)} \\ 0 & \frac{r^4+1}{2r^2} & \frac{r^6-r^4+r^2-1}{4r^2} & \frac{r^6-r^4+r^2-1}{4r^2} & 0 \\ 1 & \frac{r^4-2r^2+1}{2r^2} & \frac{r^6-r^4+r^2-1}{2r^2} & 0 & 0 \end{pmatrix}$$

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